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LETTER TO THE EDITOR

Levinson's theorem for non-local interactions†

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Abstract. Two definitions of the scattering phase shift are given. For non-local interactions giving rise to continuum bound states, the two definitions yield different phase shifts. Accordingly two different versions of Levinson's theorem result; one of them takes the continuum bound states into account, the other one does not. The former is usually used by physicists, in spite of indications that the latter corresponds more closely to physical phenomena.

1. Introduction

The first attempt at a formulation of Levinson's theorem for general scattering systems has been made by Jauch (1957) under the assumption that there are no continuum bound states (CBS). This assumption cannot usually be made for non-local interactions. Therefore Martin (1958) examined the phase shift due to a non-local, separable, central interaction in detail, and concluded that Levinson's theorem had to be modified so as to take CBS into account. An analogous result for a larger class of non-local interactions has been proved by Bertero *et al* (1968). On the other hand Buslaev (1969) has proved that Levinson's theorem holds in its original form for a class of interactions quite similar to that considered by Bertero *et al* (1968), and this in spite of the presence of CBS. In the present letter this apparent contradiction will be analysed and explained. In particular, it will be shown that both results are correct and that the discrepancy is due to a difference in the definition of the phase shift. It will be argued that Buslaev's choice of phase shift is better motivated from the point of view of physics.

For the special case of central s-wave scattering an analogous observation has already been made by Bolsterli (1969) and elaborated by Beregi *et al* (1973). It seems, however, worthwhile to discuss the matter again for the following reasons: scattering systems of a far more general nature will be considered; the perturbation is not required to be spherically symmetric and the unperturbed Hamiltonian is almost arbitrary. In addition to that it appears that most physicists are unaware of Buslaev's and Bolsterli's work; thus they continue to use the 'less physical' choice for phase shift (Englefield and Shoukry 1974, Glöckle and Le Tourneux 1976) or else rediscover the 'more physical' choice (Kermode 1976).

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2. Two Levinson-type theorems

The scattering system to be considered is given by an unperturbed Hamiltonian H and a perturbation Q. H can be the one-particle kinetic energy $H_0 = P^2/2m$ but much more general H are admissible. We only require that the spectrum of H equals the positive semi-axis $[0, \infty)$. (Note that even this requirement is imposed only for simplicity of exposition; any shape of the spectrum can be treated; for details see Dreyfus 1976.) In particular we admit $H = H_0 + V$, with V a local potential. If H has bound states, we denote their span by $\mathscr{B}$ and its orthocomplement by $\mathscr{B}^\perp$. The perturbation Q is assumed to be of the form

$$Q = gP_\phi = g|\phi\rangle\langle\phi|, \qquad g<0,\ \phi \in \mathscr{B}^\perp,\ \|\phi\| = 1;$$

that is, Q is a non-local, separable interaction; P_ϕ is the projection on the (one-dimensional) subspace spanned by ϕ; ϕ need not be spherically symmetric. The case $g>0$ can be treated as well but is less interesting because it cannot give rise to proper (negative energy) bound states.

Kato (1957) has shown that, for the scattering system given by H and Q, the scattering operator S exists and is unitary on $\mathscr{B}^\perp$ (we assume here assumption A; see the appendix); moreover S is diagonal in the energy, $\langle\lambda|S|\lambda'\rangle = \delta(\lambda - \lambda')S(\lambda)$, but not necessarily in the angular momentum. Instead of the fixed angular momentum phase shifts, one therefore considers their sum:

$$\sum (2l+1)\delta_l(\lambda)$$
$$= \frac{1}{2\mathrm{i}}\sum (2l+1) \ln \mathrm{e}^{2\mathrm{i}\delta_l(\lambda)} = \frac{1}{2\mathrm{i}}\sum \mathrm{Tr} \ln S_l(\lambda) = \frac{1}{2\mathrm{i}}\mathrm{Tr} \ln S(\lambda) = \frac{1}{2\mathrm{i}}\ln \det S(\lambda).$$

The determinant of the scattering matrix, $\det S(\lambda)$, is the object we will be concerned with mainly. Before stating its properties, we impose some not very restrictive but rather technical supplementary conditions on H and ϕ. In order not to interrupt the line of the argument, we refer these conditions, together with indications about the proofs, to the appendix.

If H and ϕ satisfy assumptions A and B of the appendix then $\det S(\lambda)$ has the following properties: there exists a function $f(z)$, holomorphic and non-zero in the open upper half-plane, with boundary value $f(\lambda) = \lim_{\delta\downarrow 0} f(\lambda + \mathrm{i}\delta)$ and zero-set $\mathscr{E} = \{\mu \in \mathbb{R} | f(\mu) = 0\}$ such that:

(*a*) $$\det S(\lambda) = \frac{f(\lambda)^*}{f(\lambda)} \qquad \text{for all } \lambda \notin \mathscr{E}. \tag{1}$$

(*b*) $\mathscr{E}$ coincides with the set of those eigenvalues of $H + Q$ which are not eigenvalues of H. This set is finite. It consists of $N_<$ proper bound state energies below zero $(0 \leqslant N_< \leqslant 1)$ and $N_>$ CBS energies above zero $(0 \leqslant N_> < \infty)$.

(*c*) $f(\lambda)$ is continuous. Moreover, for all $\mu \in \mathscr{E}$

$$\lim_{\lambda\downarrow\mu} \det S(\lambda) = \lim_{\lambda\uparrow\mu} \det S(\lambda) = 1. \tag{2}$$

Consequently, there are two obvious ways to define the logarithm of $\det S(\lambda)$ and thus the phase shift. The first makes use of the continuity property (2) of $\det S(\lambda)$ and defines the phase shift $\delta_1(\lambda)$ as the continuous logarithm of det $S(\lambda)$:

$$\delta_1(\lambda)=\frac{1}{2\mathrm{i}}\ln\det S(\lambda), \qquad \delta_1(\infty)=0.$$

($\delta_1(\infty)=0$ is the usual normalization condition; it is of no further importance here.) The second way makes use of the existence of analytic extensions and defines the phase shift $\delta_2(\lambda)$ as a boundary value:

$$\delta_2(\lambda)=\lim_{\delta\downarrow 0}-\arg f(\lambda+\mathrm{i}\delta), \qquad \delta_2(\infty)=0.$$

$\delta_1(\lambda)$ is continuous. It satisfies the modified version of Levinson's theorem proved by Martin (1958) and Bertero *et al* (1968), i.e.

$$\delta_1(0)=\pi(N_<+N_>). \tag{3}$$

$\delta_2(\lambda)$ is not continuous. More precisely, at $\mu\in\mathscr{E}$ it has a jump given by

$$\lim_{\lambda\downarrow\mu}\delta_2(\lambda)-\lim_{\lambda\uparrow\mu}\delta_2(\lambda)=\pi; \tag{4}$$

elsewhere it is continuous. It satisfies Levinson's theorem in its original form as proved by Buslaev (1969), i.e.

$$\delta_2(0)=\pi N_<. \tag{5}$$

We conclude that there is no contradiction between the results of Martin and Bertero *et al* and those of Buslaev. On the contrary, taking (4) into account, it is easy to see that (3) and (5) follow from each other and are therefore equivalent.

3. Concluding remarks

Although the phase shift or the determinant of the scattering matrix is not a directly observable quantity, one expects it to reflect the phenomena occurring in the scattering process. These phenomena do, at least within the positive energy range, supposedly depend continuously on the coupling constant. One may thus expect the phase shift to depend continuously on the coupling constant. In the case at hand continuous dependence on the coupling constant can be achieved only by admitting discontinuous dependence on the energy at CBS energies, as in (4). Isolated discontinuities in the energy seem, however, not to be a serious drawback since in practice one always deals with wave packets comprising an entire energy range. In fact such discontinuities are in perfect agreement with phenomenology; they correspond to infinitely sharp resonances and, just like the CBS they are associated with, go over into ordinary sharp resonances under slight changes of the coupling constant. In this sense the phase shift $\delta_2(\lambda)$ can be said to be 'more physical' than $\delta_1(\lambda)$. Numerical examples as well as sketches illustrating this discussion can be found in the papers by Beregi *et al* (1973) and Kermode (1976).

Finally we would like to mention that the present formulation of Levinson's theorem in an abstract framework has led to an analogous theorem of a far more general nature (Dreyfus 1976); this theorem in turn can be applied to provide Levinson-type theorems for scattering by local but non-central potentials (Dreyfus 1975) and for scattering by impurities in crystals (Dreyfus, in preparation).

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Appendix

Here we collect some technical assumptions about the scattering system and indicate how the statements in § 2 are proved. For complete proofs the reader is referred to either §§ 4 and 5 of Dreyfus (1976) or §§ E3 and E4 of Dreyfus (1975).

Assumption A. H has no singular continuous spectrum. (No physical Hamiltonian is known to have singular continuous spectrum.)

Assumption B. For $f(z) = 1 + g\langle\phi|(H-z)^{-1}|\phi\rangle$

(B1) $\lim_{\delta\downarrow 0} f(\lambda + i\delta)$ exists for all λ and is continuous in λ;
(B2) $f(\lambda)$ is differentiable at all $\lambda \in \mathscr{E}$;
(B3) $|f(\lambda) - 1| < c(1+\lambda)^{-\theta}$ for some $c > 0$ and $\theta > \frac{1}{2}$ and for all $\lambda > 0$.

Assumption B imposes some smoothness and some decrease properties on the momentum space representative of ϕ. The set of ϕ satisfying these properties is very large; in particular it is dense in $\mathscr{B}^{\perp}$.

Outline of proofs. Formula (1) follows from the representation for S given by Kuroda (1963). The f of formula (1) is identical with the f of assumption B; it is the Fredholm determinant of the scattering system. This implies that $\mathscr{E}$ coincides with the set of those eigenvalues of $H+Q$, which are not eigenvalues of H. Assumption B is essential for $\mathscr{E}$ to be finite. Details may be found in a paper by Faddeev (1967). Formula (2) follows from the Bernoulli–L'Hôpital rule, applied to (1).

We give no separate proof of (3) because it is easier to prove (4) and (5), from which (3) follows. The proof of (4) and (5) is based on the technical result that at every zero of f the derivative of f is strictly negative. (Note that f is a complex-valued function !) Using this, together with $\mathrm{Im}\, f \geq 0$ for all z and $\mathrm{Im}\, f(\lambda) = 0$ for $\lambda \leq 0$, it is not difficult to see that $\arg f(\lambda)$ does not vary as λ runs from $-\infty$ to the smallest zero of f, but that it decreases by π as λ runs from any zero of f to the next or from the largest zero of f to $+\infty$. (Note that $f(\lambda) \to 1$ as $\lambda \to \pm\infty$.) Similarly $\arg f(\lambda)$ increases by π as λ passes through any zero of f. Thus (4) holds. (5) results by summing up the above variations.

References

Beregi P, Zakhar'ev B N and Niyazgulov S A 1973 *Sov. J. Part. Nucl.* **4** 217–30
Bertero M, Talenti G and Viano G A 1968 *Nucl. Phys.* A **113** 625–40
Bolsterli M 1969 *Phys. Rev.* **182** 1095–6
Buslaev V S 1969 *Probl. Math. Phys. (Leningrad)* **4** 43–53
Dreyfus T 1975 *PhD Thesis* Université de Genève
—— 1976 *J. Math. Anal. Appl.* to be published
Englefield M J and Shoukry H S M 1974 *Prog. Theor. Phys.* **52** 1554–66
Faddeev L D 1967 *Am. Math. Soc. Transl.* **62** 177–203

Glöckle W and Le Tourneux J 1976 *Nucl. Phys.* A **269** 16–30
Jauch J M 1957 *Helv. Phys. Acta* **31** 127–58
Kato T 1957 *J. Math. Soc. Japan* **9** 239–49
Kermode M W 1976 *J. Phys. A: Math. Gen.* **9** L47–9
Kuroda S T 1963 *Pacific J. Math.* **13** 1305–18
Martin A 1958 *Nuovo Cim.* **7** 607–27